Integral Transforms and Special Functions

ISSN: 1065-2469 (Print) 1476-8291 (Online) Journal homepage: http://www.tandfonline.com/loi/gitr20

Properties of the fractional (exponential) Radon transform

Sunghwan Moon

INTEGRAL TRANSFORMS AND SPECIAL FUNCTIONS, 2017
VOL. 28, NO. 12, 923–939
https://doi.org/10.1080/10652469.2017.1390666

Research Article

Properties of the fractional (exponential) Radon transform

Sunghwan Moon

Department of Mathematics, Kyungpook National University, Daegu, Republic of Korea

ABSTRACT
The fractional Radon transform defined, based on the Fourier slice theorem and the fractional Fourier transform, has many potential applications in optics and the pattern-recognition field. Here we study many properties of the fractional Radon transform using existing theory of the regular Radon transform: the inversion formulas, stability estimates, uniqueness and reconstruction for a local data problem, and a range description. Also, we define the fractional exponential Radon transform and present its inversion.

ARTICLE HISTORY
Received 6 March 2017
Accepted 5 October 2017

KEYWORDS
Fractional; Fourier transform; Radon transform; exponential Radon transform; tomography; X-ray transform

AMS CLASSIFICATION
44A12; 42A38; 92C55

1. Introduction

The fractional Fourier transform was first introduced in the 1980s in a purely abstract way as a novel method of solving the Schrödinger equation under various conditions [1,2]. However, this fractional Fourier transform had not brought much attention until Mendlovic and Ozaktas physically interpreted it with relation to quadratic graded-index media (GRIN media) in [3]. After that, many properties and optical implementations of the fractional Fourier transform were studied in [4–10]. (In [10], the authors called the fractional Fourier transform the Radon–Wigner transform and showed its many applications in optics.) Recently, application to image reconstruction in MRI was suggested in [11].

The Radon transform is also a promising tool in optical signal-processing systems and is very related to the Fourier transform. Using the Fourier slice theorem, the two-dimensional fractional Radon transform was first introduced by Zalevsky and Mendlovic [12] in 1996. They mentioned its basic properties such as linearity, rotation invariance, shift, and the Fourier slice theorem, and its many applications in optics as well as the pattern-recognition field. In particular, in [13] its possible applications were illustrated: the minimization of the mean-square error obtained after filtering non-stationary signals is directly related to the fractional Radon transform. In [14], the n-dimensional fractional Radon transform was introduced and its inversion formula was derived. Here we study many properties of the fractional Radon transform using the existing results for the regular Radon transform (see, e.g. [15–17]). Also, we introduce the fractional X-ray transform and present its

CONTACT Sunghwan Moon sunghwan.moon@knu.ac.kr Department of Mathematics, Kyungpook National University, Daegu 41566, Republic of Korea

inversion formula and stability estimates. Lastly, we introduce the fractional exponential Radon transform and derive its inversion.

In Section 2, we briefly introduce the fractional Fourier transform of order α and its basic properties. In Section 3, the fractional Radon and X-ray transforms are defined and their elementary properties are studied. Section 3.1 is devoted to the inversion formulas which is similar to that for the regular Radon and X-ray transforms in [17, Theorems 2.1, 2.2, and 2.3 in Chapter II]. In Section 3.2, we show that taking a certain linear operator on the fractional Radon transform is an isometry on the new space with the defined norm based on the fractional Fourier transform and discuss stability estimates. In Section 3.3, we study the uniqueness and reconstruction for a limited data and describe the range conditions. Section 4 is devoted to the introduction and inversion of the fractional exponential Radon transform.

2. Preliminary

For $\alpha \in \mathbb{R}$ and an integrable function f, the fractional Fourier transform $\mathcal{F}_\alpha f$ of f of order α is defined by

$$\mathcal{F}_\alpha f(\boldsymbol{\xi}) = \begin{cases} (C_\alpha)^n \displaystyle\int_{\mathbb{R}^n} e^{i2^{-1}(|\boldsymbol{\xi}|^2+|\mathbf{x}|^2)\cot\alpha - i\boldsymbol{\xi}\cdot\mathbf{x}\csc\alpha} f(\mathbf{x})\,d\mathbf{x} & \text{if } \alpha \text{ is not a multiple of } \pi, \\ f(\boldsymbol{\xi}) & \text{if } \alpha \text{ is a multiple of } 2\pi, \\ f(-\boldsymbol{\xi}) & \text{if } \alpha+\pi \text{ is a multiple of } 2\pi, \end{cases} \tag{2.1}$$

where

$$C_\alpha = \left(\frac{1 - i\cot\alpha}{2\pi}\right)^{1/2}.$$

Clearly, $\mathcal{F}_{\pi/2} f = \mathcal{F}f$ is the regular Fourier transform of f. Many properties of the fractional Fourier transform were introduced in [1,4,7–9]:

- Linearity: $\mathcal{F}_\alpha[c_1 f_1(\mathbf{x}) + c_2 f_2(\mathbf{x})](\boldsymbol{\xi}) = c_1\mathcal{F}_\alpha f_1(\boldsymbol{\xi}) + c_2\mathcal{F}_\alpha f_2(\boldsymbol{\xi})$.
- Inverse: $\mathcal{F}_{-\alpha}\mathcal{F}_\alpha f = f$.
- Parseval identity:

$$\int_{\mathbb{R}^n} f(\mathbf{x})[g(\mathbf{x})]^- d\mathbf{x} = \int_{\mathbb{R}^n} \mathcal{F}_\alpha f(\boldsymbol{\xi})[\mathcal{F}_\alpha g(\boldsymbol{\xi})]^- d\boldsymbol{\xi}, \tag{2.2}$$

where z^- is the complex conjugate of the complex number z.

Notice that

$$(C_\alpha)^{-n}\mathcal{F}_\alpha f(\boldsymbol{\xi}) = (2\pi)^{n/2} e^{i2^{-1}|\boldsymbol{\xi}|^2\cot\alpha}\mathcal{F}[e^{i2^{-1}|\mathbf{x}|^2\cot\alpha} f(\mathbf{x})](\boldsymbol{\xi}\csc\alpha). \tag{2.3}$$

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space consisting of C^∞ functions which, together with all their partial derivatives, vanish at infinity faster than any power of $|\mathbf{x}|$.

Theorem 2.1 (Theorem 3.1 in [1]): *The fractional Fourier transform $\mathcal{F}_\alpha$ is homeomorphism on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ with inverse $\mathcal{F}_{-\alpha}$.*

3. The fractional Radon and X-ray transforms

For $0 < |\alpha| < \pi$, we define the fractional Radon transform R_α and the fractional X-ray transform X_α by

$$\begin{aligned} R_\alpha f(\mathbf{e}_{\boldsymbol\theta}, s) &= \int_{\mathbf{e}_{\boldsymbol\theta}^{\perp}} f(s\mathbf{e}_{\boldsymbol\theta} + \boldsymbol\tau)\, \mathrm{e}^{\mathrm{i}2^{-1}|\boldsymbol\tau|^2 \cot\alpha}\, \mathrm{d}\boldsymbol\tau \\ &= \int_{\mathbb{R}^n} f(\mathbf{x})\, \mathrm{e}^{\mathrm{i}2^{-1}(|\mathbf{x}|^2 - s^2)\cot\alpha}\, \delta(\mathbf{e}_{\boldsymbol\theta}\cdot \mathbf{x} - s)\, \mathrm{d}\mathbf{x}, \end{aligned} \tag{3.1}$$

where δ is the one-dimensional distribution function [14] and $\mathbf{e}_{\boldsymbol\theta} \in S^{n-1}$ is the unit vector, and

$$X_\alpha f(\mathbf{e}_{\boldsymbol\theta}, \mathbf{y}) = \int_{\mathbb{R}} f(\tau\mathbf{e}_{\boldsymbol\theta} + \mathbf{y})\, \mathrm{e}^{\mathrm{i}2^{-1}|\tau|^2\cot\alpha}\, \mathrm{d}\tau, \quad \text{for } \mathbf{y} \in \mathbf{e}_{\boldsymbol\theta}^{\perp}.$$

When $\alpha = \pi/2$, $R_\alpha f$ and $X_\alpha f$ are the regular Radon and X-ray transforms, respectively. Notice that $R_\alpha f$ is even, i.e. $R_\alpha f(\mathbf{e}_{\boldsymbol\theta}, s) = R_\alpha f(-\mathbf{e}_{\boldsymbol\theta}, -s)$ and

$$R_\alpha f(\mathbf{e}_{\boldsymbol\theta}, s) = \mathrm{e}^{-\mathrm{i}2^{-1}s^2\cot\alpha} R(\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f(\mathbf{x}))(\mathbf{e}_{\boldsymbol\theta}, s), \tag{3.2}$$

$$X_\alpha f(\mathbf{e}_{\boldsymbol\theta}, \mathbf{y}) = \mathrm{e}^{-\mathrm{i}2^{-1}|\mathbf{y}|^2\cot\alpha} X(\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f(\mathbf{x}))(\mathbf{e}_{\boldsymbol\theta}, \mathbf{y}). \tag{3.3}$$

Also, we have for any $\mathbf{e}_{\boldsymbol\theta} \in S^{n-1}$ with $\mathbf{e}_{\boldsymbol\theta} \perp \mathbf{e}_{\boldsymbol\omega}$,

$$R_\alpha f(\mathbf{e}_{\boldsymbol\omega}, s) = \int_{\mathbf{y}\in\mathbf{e}_{\boldsymbol\theta}^{\perp}, \mathbf{y}\cdot\mathbf{e}_{\boldsymbol\omega} = s} X_\alpha f(\mathbf{e}_{\boldsymbol\theta}, \mathbf{y})\, \mathrm{d}\mathbf{y},$$

like the regular Radon and X-ray transforms (see [17, Equation(1.1) on the p.10]).

Theorem 3.1: *For $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < |\alpha| < \pi$, we have*

$$\begin{aligned} \mathcal{F}_{\alpha,s}(R_\alpha f)(\mathbf{e}_{\boldsymbol\theta}, \sigma) &= (C_\alpha)^{1-n} \mathcal{F}_\alpha f(\sigma \mathbf{e}_{\boldsymbol\theta}), \\ \mathcal{F}_{\alpha,\mathbf{y}}(X_\alpha f)(\mathbf{e}_{\boldsymbol\theta}, \boldsymbol\eta) &= (C_\alpha)^{-1} \mathcal{F}_\alpha f(\boldsymbol\eta), \quad \boldsymbol\eta \in \mathbf{e}_{\boldsymbol\theta}^{\perp}, \end{aligned}$$

where $\mathcal{F}_{\alpha,s}$ and $\mathcal{F}_{\alpha,\mathbf{y}}$ are the one-dimensional and $n-1$-dimensional fractional Fourier transform operators with respect to the variable s and $\mathbf{y}$, respectively.

In fact, the fractional Radon transform was defined based on Theorem 3.1 [12,14].

Proof: Taking the fractional Fourier transform of $R_\alpha f$ with respect to s, we have

$$\mathcal{F}_{\alpha,s}(R_\alpha f)(\mathbf{e}_{\boldsymbol\theta}, \sigma) = C_\alpha \int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}2^{-1}(\sigma^2 + |\mathbf{x}|^2)\cot\alpha - \mathrm{i}\mathbf{x}\cdot\mathbf{e}_{\boldsymbol\theta}\sigma\csc\alpha} f(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

Definition (2.1) of the fractional Fourier transform of f completes our proof.

Similarly, we have

$$\begin{aligned}\mathcal{F}_{\alpha,\mathbf{y}}(X_\alpha f)(\mathbf{e}_\theta,\boldsymbol{\eta}) &= C_\alpha^{n-1}\int_{\mathbf{e}_\theta^\perp} e^{i2^{-1}(|\boldsymbol{\eta}|^2+|\mathbf{y}|^2)\cot\alpha - i\boldsymbol{\eta}\cdot\mathbf{y}\csc\alpha} X_\alpha f(\tau\mathbf{e}_\theta+\mathbf{y})\,d\mathbf{y}\\ &= C_\alpha^{n-1}\int_{\mathbf{e}_\theta^\perp} e^{i2^{-1}(|\boldsymbol{\eta}|^2+|\mathbf{y}|^2)\cot\alpha - i\boldsymbol{\eta}\cdot\mathbf{y}\csc\alpha}\int_{\mathbb{R}} f(\tau\mathbf{e}_\theta+\mathbf{y})\,e^{i2^{-1}|\tau|^2\cot\alpha}\,d\tau\,d\mathbf{y}\\ &= C_\alpha^{n-1}\int_{\mathbb{R}^n} e^{i2^{-1}(|\boldsymbol{\eta}|^2+|\mathbf{x}|^2)\cot\alpha - i\boldsymbol{\eta}\cdot\mathbf{x}\csc\alpha} f(\mathbf{x})\,d\mathbf{x},\end{aligned}$$

where in the last line, we changed the variables $\mathbf{x}=\mathbf{y}+\tau\mathbf{e}_\theta$ so that $|\mathbf{x}|^2=|\mathbf{y}|^2+|\tau|^2$. Again, definition (2.1) completes our proof. ■

If $f\in\mathcal{S}(\mathbb{R}^n)$, then $e^{i2^{-1}|\mathbf{x}|^2\cot\alpha}f\in\mathcal{S}(\mathbb{R}^n)$ and thus $R_\alpha f$ is in the Schwartz class on $S^{n-1}\times\mathbb{R}$ defined by restricting the functions in $\mathcal{S}(\mathbb{R}^{n+1})$ to $S^{n-1}\times\mathbb{R}$.

Now we introduce the backprojection operators $R_\alpha^\#$ and $X_\alpha^\#$ by

$$\begin{aligned}R_\alpha^\# g(\mathbf{x}) &= \int_{S^{n-1}} g(\mathbf{e}_\theta,\mathbf{x}\cdot\mathbf{e}_\theta)\,e^{-i2^{-1}(|\mathbf{x}|^2-(\mathbf{e}_\theta\cdot\mathbf{x})^2)\cot\alpha}\,dS(\mathbf{e}_\theta),\\ X_\alpha^\# g(\mathbf{x}) &= \int_{S^{n-1}} g(\mathbf{e}_\theta,E_{\mathbf{e}_\theta}(\mathbf{x}))\,e^{-i2^{-1}(\mathbf{e}_\theta\cdot\mathbf{x})^2\cot\alpha}\,dS(\mathbf{e}_\theta).\end{aligned}$$

for $g\in C^\infty(S^{n-1}\times\mathbb{R})$ or $g\in C^\infty(T)$ with compact support. Here $T=\{(\mathbf{e}_\theta,\mathbf{y}):\mathbf{e}_\theta\in S^{n-1},\mathbf{y}\in\mathbf{e}_\theta^\perp\}$, $E_{\mathbf{e}_\theta}$ is the orthogonal projection on $\mathbf{e}_\theta^\perp$ and $dS(\mathbf{e}_\theta)$ is the standard area measure on the unit sphere in $\mathbb{R}^n$. Then $R_\alpha^\#$ and $X_\alpha^\#$ are the dual operators to R_α and X_α, respectively. In fact, for $g\in C^\infty(S^{n-1}\times\mathbb{R})$ and $f\in C^\infty(\mathbb{R}^n)$ with compact support, we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{S^{n-1}} g(\mathbf{e}_\theta,s)[R_\alpha f(\mathbf{e}_\theta,s)]^- dS(\mathbf{e}_\theta)\,ds\\ &= \int_{\mathbb{R}}\int_{S^{n-1}} g(\mathbf{e}_\theta,s)\int_{\mathbb{R}^n}[f(\mathbf{x})]^- e^{-i2^{-1}(|\mathbf{x}|^2-s^2)\cot\alpha}\delta(\mathbf{e}_\theta\cdot\mathbf{x}-s)\,d\mathbf{x}\,dS(\mathbf{e}_\theta)\,ds\\ &= \int_{\mathbb{R}^n} R_\alpha^\# g(\mathbf{x})[f(\mathbf{x})]^- d\mathbf{x},\end{aligned}$$

where z^- is the complex conjugate of the complex number z, again. Similarly, we can show

$$\int_{S^{n-1}}\int_{\mathbf{e}_\theta^\perp} g(\mathbf{e}_\theta,\mathbf{y})[X_\alpha f(\mathbf{e}_\theta,\mathbf{y})]^- d\mathbf{y}\,dS(\mathbf{e}_\theta) = \int_{\mathbb{R}^n} X_\alpha^\# g(\mathbf{x})[f(\mathbf{x})]^- d\mathbf{x}.$$

Notice that

$$R_\alpha^\# g(\mathbf{x}) = e^{-i2^{-1}|\mathbf{x}|^2\cot\alpha} R^\#(e^{i2^{-1}s^2\cot\alpha}g)(\mathbf{x}), \tag{3.4}$$

$$X_\alpha^\# g(\mathbf{x}) = e^{-i2^{-1}|\mathbf{x}|^2\cot\alpha} X^\#(e^{i2^{-1}|\mathbf{y}|^2\cot\alpha}g)(\mathbf{x}), \tag{3.5}$$

where $R^\# = R_{\pi/2}^\#$ and $X^\# = X_{\pi/2}^\#$ are the backprojection operators of the regular Radon and X-ray transforms, respectively.

Proposition 3.2: *For $g \in \mathcal{S}(S^{n-1} \times \mathbb{R})$ and $0 < |\alpha| < \pi$, we have*

$$\mathcal{F}_\alpha(R_\alpha^\# g)(\boldsymbol{\xi}) = (C_\alpha)^{1-n}|\boldsymbol{\xi}|^{1-n}\left(\mathcal{F}_{\alpha,s}g\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}, |\boldsymbol{\xi}|\right) + \mathcal{F}_{\alpha,s}g\left(-\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}, -|\boldsymbol{\xi}|\right)\right).$$

This proof is similar to the proof of Theorem 1.4 in Chapter 1 in [17].

3.1. Inversion formulas

For $\gamma < n$, we define the linear operator I_α^γ by

$$\mathcal{F}_\alpha(I_\alpha^\gamma f)(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-\gamma}\mathcal{F}_\alpha f(\boldsymbol{\xi}). \tag{3.6}$$

For $f \in \mathcal{S}(\mathbb{R}^n)$, $\mathcal{F}_\alpha(I_\alpha^\gamma f)(\boldsymbol{\xi}) = |\boldsymbol{\xi}|^{-\gamma}\mathcal{F}_\alpha f(\boldsymbol{\xi}) \in L^1(\mathbb{R}^n)$, hence $I_\alpha^\gamma f$ makes sense. When $\alpha = \pi/2$, $I_{\pi/2}^\gamma = I^\gamma$ is the regular Riesz potential.

Lemma 3.3: *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < |\alpha| < \pi$. For $\gamma < n$, we have*

$$|\csc\alpha|^\gamma I^\gamma(\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f)(\mathbf{x}) = \mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} I_\alpha^\gamma f(\mathbf{x}).$$

Proof: Taking the Fourier transform of $I^\gamma(\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f)(\mathbf{x})$ yields

$$\begin{aligned}
\mathcal{F}[|\csc\alpha|^\gamma I^\gamma(\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f)](\boldsymbol{\xi}\csc\alpha) &= |\boldsymbol{\xi}|^{-\gamma}\mathcal{F}[\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f](\boldsymbol{\xi}\csc\alpha)\\
&= (2\pi)^{-n/2}(C_\alpha)^{-n}\,\mathrm{e}^{-\mathrm{i}2^{-1}|\boldsymbol{\xi}|^2\cot\alpha}|\boldsymbol{\xi}|^{-\gamma}\mathcal{F}_\alpha f(\boldsymbol{\xi})\\
&= (2\pi)^{-n/2}(C_\alpha)^{-n}\,\mathrm{e}^{-\mathrm{i}2^{-1}|\boldsymbol{\xi}|^2\cot\alpha}\mathcal{F}_\alpha[I_\alpha^\gamma f(\mathbf{x})](\boldsymbol{\xi})\\
&= \mathcal{F}[\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} I_\alpha^\gamma f(\mathbf{x})](\boldsymbol{\xi}\csc\alpha),
\end{aligned}$$

where in the second and fourth equalities, we used the identity (2.3). ■

Theorem 3.4: *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $0 < |\alpha| < \pi$. For $\gamma < n$, we have*

$$f(\mathbf{x}) = 2^{-1}(2\pi)^{1-n}|\csc\alpha|^{n-1}[I_\alpha^{-\gamma}R_\alpha^\# I_\alpha^{\gamma-n+1}(R_\alpha f)](\mathbf{x}), \tag{3.7}$$

$$f(\mathbf{x}) = \frac{(2\pi)^{-1}}{|S^{n-2}|}|\csc\alpha|[I_\alpha^{-\gamma}X_\alpha^\# I_\alpha^{\gamma-1}(X_\alpha f)](\mathbf{x}). \tag{3.8}$$

When $\gamma = 0$, (3.7) becomes

$$\begin{aligned}
f(\mathbf{x}) &= 2^{-1}(2\pi)^{1-n}|\csc\alpha|^{n-1}R_\alpha^\#[I_\alpha^{-n+1}(R_\alpha f)](\mathbf{x})\\
&= (2\pi)^{-n}|\csc\alpha|^n\int_{\mathbb{R}}\int_0^\infty\int_{S^{n-1}}(R_\alpha f)(\mathbf{e}_{\boldsymbol{\theta}}, s)\sigma^{n-1}\\
&\quad\times \mathrm{e}^{\mathrm{i}2^{-1}(s^2-|\mathbf{x}|^2)\cot\alpha - \mathrm{i}\sigma(s-\mathbf{x}\cdot\mathbf{e}_{\boldsymbol{\theta}})\csc\alpha}\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}})\,\mathrm{d}\sigma\,\mathrm{d}s,
\end{aligned}$$

which is equivalent to (33) derived in [14].

Proof: We notice that for $\gamma < n$,

$$f(\mathbf{x}) = 2^{-1}(2\pi)^{1-n}I^{-\gamma}R^\# I^{\gamma-n+1}Rf(\mathbf{x}) \tag{3.9}$$

(see [17, Theorem 2.1 in Chapter II]). Together with (3.2), we have

$$f(\mathbf{x}) = 2^{-1}(2\pi)^{1-n}\,\mathrm{e}^{-\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} I^{-\gamma}[R^{\#} I^{\gamma-n+1}(\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha} R_\alpha f)](\mathbf{x}). \tag{3.10}$$

By Lemma 3.3 and (3.4), we have

$$\begin{aligned} f(\mathbf{x}) &= 2^{-1}(2\pi)^{1-n}|\csc\alpha|^{-\gamma+n-1}\,\mathrm{e}^{-\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha}[I^{-\gamma} R^{\#}\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha} I_\alpha^{\gamma-n+1}(R_\alpha f)](\mathbf{x}) \\ &= 2^{-1}(2\pi)^{1-n}|\csc\alpha|^{-\gamma+n-1}\,\mathrm{e}^{-\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha}[I^{-\gamma}\,\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} R_\alpha^{\#} I_\alpha^{\gamma-n+1}(R_\alpha f)](\mathbf{x}) \\ &= 2^{-1}(2\pi)^{1-n}|\csc\alpha|^{n-1}[I_\alpha^{-\gamma} R_\alpha^{\#} I_\alpha^{\gamma-n+1}(R_\alpha f)](\mathbf{x}). \end{aligned}$$

For the second inversion formula, we do the above process with (3.3), (3.5), and

$$f(\mathbf{x}) = \frac{(2\pi)^{-1}}{|S^{n-2}|}[I^{-\gamma} X^{\#} I^{\gamma-1}(Xf)](\mathbf{x}) \quad \text{(see [17, Theorem 2.1 in Chapter II])}$$

instead of (3.2), (3.4) and (3.9), respectively. ■

From the inversion formula in Theorem 3.4, we obtain the following version of the Plancherel formula:

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{x})[g(\mathbf{x})]^{-}\mathrm{d}\mathbf{x} &= \frac{|\csc\alpha|^{n-1}}{2(2\pi)^{n-1}}\int_{\mathbb{R}^n} R_\alpha^{\#} I^{1-n}(R_\alpha f)(\mathbf{x})[g(\mathbf{x})]^{-}\mathrm{d}\mathbf{x} \\ &= \frac{|\csc\alpha|^{n-1}}{2(2\pi)^{n-1}}\int_{S^{n-1}}\int_{\mathbb{R}} I^{1-n}(R_\alpha f)(\mathbf{e}_\theta,s)[R_\alpha g(\mathbf{e}_\theta,s)]^{-}\mathrm{d}s\,\mathrm{d}S(\mathbf{e}_\theta). \end{aligned}$$

Here in the last line, we used the definition of $R_\alpha^{\#}$ and the such formula for the regular Radon transform is derived in [16]. Similarly, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} f(\mathbf{x})[g(\mathbf{x})]^{-}\mathrm{d}\mathbf{x} &= \frac{(2\pi)^{-1}}{|S^{n-2}|}|\csc\alpha|\int_{\mathbb{R}^n} X_\alpha^{\#} I^{-1}(X_\alpha f)(\mathbf{x})[g(\mathbf{x})]^{-}\mathrm{d}\mathbf{x} \\ &= \frac{(2\pi)^{-1}}{|S^{n-2}|}|\csc\alpha|\int_{S^{n-1}}\int_{\mathbf{e}_\theta^{\perp}} I^{-1}(X_\alpha f)(\mathbf{e}_\theta,\mathbf{y})[X_\alpha g(\mathbf{e}_\theta,\mathbf{y})]^{-}\mathrm{d}\mathbf{y}\,\mathrm{d}S(\mathbf{e}_\theta). \end{aligned}$$

From (3.10) with $\gamma = 0$ and $I^{1-n} = (-1)^{(n-1)/2}\Delta^{(n-1)/2}$, we have the following observation: For n odd, the problem of reconstructing a function from the fractional Radon transform is still local, like the regular Radon transform case. Here 'local' means that the function is determined at a point by the fractional Radon transform on a neighbourhood of that point.

Let us expand f and $g_\alpha = R_\alpha f$ in spherical harmonics:

$$f(\mathbf{x}) = \sum_{l=0}^{\infty}\sum_{k=0}^{N(n,l)} f_{lk}(|\mathbf{x}|)Y_{lk}(\mathbf{x}/|\mathbf{x}|), \quad g_\alpha(\mathbf{e}_\theta,s) = \sum_{l=0}^{\infty}\sum_{k=0}^{N(n,l)} g_{\alpha,lk}(s)Y_{lk}(\mathbf{e}_\theta),$$

where $N(n,l) = (2l+n-2)(n+l-3)!/l!(n-2)!$, $N(n,0) = 1$, and $Y_{lk}(\mathbf{e}_\theta)$ is the spherical harmonics. For the regular Radon transform (i.e. $\alpha = \pi/2$), we know the relations between f_{lk} and $g_{\pi/2,lk}$ (see [17, Theorems 2.2 and 2.3 in Chapter II]):

$$g_{\pi/2,lk}(s) = |S^{n-2}| \int_s^\infty C_l^{(n-2)/2}\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)^{(n-3)/2} f_{lk}(r) r^{n-2}\,\mathrm{d}r,$$
$$f_{lk}(r) = c(n) \int_r^\infty C_l^{(n-2)/2}\left(\frac{s}{r}\right)(s^2-r^2)^{(n-3)/2} \partial_s^{n-1} g_{lk}(s)\mathrm{d}s,$$

where $C_l^\lambda, \lambda > -1/2$, are the Gegenbauer polynomials of degree l and

$$c(n) = \frac{(-1)^{n-1}}{2\pi^{n/2}} \frac{\Gamma((n-2)/2)}{\Gamma(n-2)}, \quad c(2) = -\pi^{-1}.$$

Together with (3.2), we have the following inversions:

Theorem 3.5: *Let $f \in C^\infty(\mathbb{R}^n)$ have compact support and $0 < |\alpha| < \pi$. Then we have*

$$g_{\alpha,lk}(s) = |S^{n-2}|\, \mathrm{e}^{-\mathrm{i}2^{-1}s^2 \cot\alpha} \int_s^\infty C_l^{(n-2)/2}\left(\frac{s}{r}\right)\left(1-\frac{s^2}{r^2}\right)^{(n-3)/2} \mathrm{e}^{\mathrm{i}2^{-1}r^2\cot\alpha} f_{lk}(r) r^{n-2}\,\mathrm{d}r,$$
$$f_{lk}(r) = c(n)\, \mathrm{e}^{-\mathrm{i}2^{-1}r^2\cot\alpha} \int_r^\infty C_l^{(n-2)/2}\left(\frac{s}{r}\right)(s^2-r^2)^{(n-3)/2} \partial_s^{n-1}[\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha} g_{\alpha,lk}](s)\,\mathrm{d}s.$$

3.2. Isometry and stability estimates

Let $L^2(\mathbb{R}^n)$ and $L^2(S^{n-1}\times\mathbb{R})$ be the regular L^2-spaces. For $\gamma \ge 0$, let $H_\alpha^\gamma(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|_{\alpha,\gamma} < \infty\}$ and $H_\alpha^\gamma(S^{n-1}\times\mathbb{R}) = \{g \in L^2(S^{n-1}\times\mathbb{R}) : \|g\|_{\alpha,\gamma} < \infty\}$ be the collections of L^2-functions bounded by the following norms:

$$\|f\|_{\alpha,\gamma}^2 = \int_{\mathbb{R}^n} |\mathcal{F}_\alpha f(\boldsymbol{\xi})|^2 (1+|\boldsymbol{\xi}|^2)^\gamma\,\mathrm{d}\boldsymbol{\xi},$$
$$\|g\|_{\alpha,\gamma}^2 = \int_{S^{n-1}}\int_{\mathbb{R}} |\mathcal{F}_{\alpha,s} g(\mathbf{e}_\theta,\sigma)|^2 (1+|\sigma|^2)^\gamma\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_\theta). \tag{3.11}$$

In particular, $\|\cdot\|_{\alpha,0}$ is the regular L^2-norm by Parseval identity (2.2) and $\|\cdot\|_{\pi/2,\gamma}$ is the regular Sobolev norm.

Let $f \in H_\alpha^\gamma(\mathbb{R}^n)$. Then $\mathcal{F}_\alpha f$ is a L^2-function with weight $(1+|\boldsymbol{\xi}|^2)^\gamma$. Thus there is a sequence $\bar{f}_n \in \mathcal{S}(\mathbb{R}^n)$ such that $\|\mathcal{F}_{-\alpha}\bar{f}_n - f\|_{\alpha,\gamma}$ goes to zero. Set $f_n = \mathcal{F}_{-\alpha}\bar{f}_n$. Then $f_n \in \mathcal{S}(\mathbb{R}^n)$ by Theorem 2.1 and $\|f_n - f\|_{\alpha,\gamma}$ goes to zero. Thus $\mathcal{S}(\mathbb{R}^n)$ is dense in $H_\alpha^\gamma(\mathbb{R}^n)$ with respect to $\|f\|_{\alpha,\gamma}$.

Also, $H_\alpha^\gamma(\mathbb{R}^n)$ and $H_\alpha^\gamma(S^{n-1}\times\mathbb{R})$ are Hilbert spaces. In particular, $H_\alpha^0(\mathbb{R}^n)$ and $H_\alpha^0(S^{n-1}\times\mathbb{R})$ are the regular L^2 spaces and $H_{\pi/2}^\gamma(\mathbb{R}^n)$ and $H_{\pi/2}^\gamma(S^{n-1}\times\mathbb{R})$ become the regular Sobolev spaces.

Theorem 3.6: *For $\gamma \ge 0$ and $0 < |\alpha| < \pi$, the mapping $f \to I_\alpha^{-(n-1)/2} R_\alpha f$ extends to an isometry of $H_\alpha^\gamma(\mathbb{R}^n)$ into $H_{\alpha,e}^\gamma(S^{n-1}\times\mathbb{R})$, where $H_{\alpha,e}^\gamma(S^{n-1}\times\mathbb{R}) = \{g \in H_\alpha^\gamma(S^{n-1}\times\mathbb{R}) : g(\mathbf{e}_\theta, s) = g(-\mathbf{e}_\theta, -s)\}$.*

Proof: For $f \in \mathcal{S}(\mathbb{R}^n)$, we start with $\|f\|^2_{\alpha,\gamma}$:

$$\begin{aligned}\|f\|^2_{\alpha,\gamma} &= \int_{\mathbb{R}^n} |\mathcal{F}_\alpha f(\boldsymbol{\xi})|^2(1+|\boldsymbol{\xi}|)^\gamma \,\mathrm{d}\boldsymbol{\xi} = \int_{S^{n-1}}\int_0^\infty |\mathcal{F}_\alpha f(\sigma \mathbf{e}_{\boldsymbol{\theta}})|^2(1+\sigma^2)^\gamma \sigma^{n-1}\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}})\\ &= 2^{-1}\int_{S^{n-1}}\int_{\mathbb{R}} |\mathcal{F}_\alpha f(\sigma \mathbf{e}_{\boldsymbol{\theta}})|^2(1+|\sigma|^2)^\gamma |\sigma|^{n-1}\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}})\\ &= 2^{-1}|C_\alpha|^{2(n-1)}\int_{S^{n-1}}\int_{\mathbb{R}} |\mathcal{F}_{\alpha,s}(R_\alpha f)(\mathbf{e}_{\boldsymbol{\theta}},\sigma)|^2(1+|\sigma|^2)^\gamma |\sigma|^{n-1}\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}})\\ &= 2^{-1}|C_\alpha|^{2(n-1)}\|I_\alpha^{-(n-1)/2}R_\alpha f\|^2_{\alpha,\gamma}, \end{aligned} \tag{3.12}$$

where in the second equality, we changed the variables $\boldsymbol{\xi} \to \sigma \mathbf{e}_{\boldsymbol{\theta}}$ and in the fourth equality, we used Theorem 3.1. It remains to prove that the mapping is surjective. It is enough to show that if $g \in H^\gamma_{\alpha,e}(S^{n-1}\times\mathbb{R})$ satisfies

$$\int_{S^{n-1}}\int_{\mathbb{R}} \mathcal{F}_{\alpha,s}g(\mathbf{e}_{\boldsymbol{\theta}},\sigma)\mathcal{F}_{\alpha,s}[I_\alpha^{-(n-1)/2}R_\alpha f](\mathbf{e}_{\boldsymbol{\theta}},\sigma)(1+|\sigma|^2)^\gamma\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}}) = 0$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$, then $g=0$. By Theorem 3.1, we have

$$\begin{aligned}0 &= \int_{S^{n-1}}\int_{\mathbb{R}} \mathcal{F}_{\alpha,s}g(\mathbf{e}_{\boldsymbol{\theta}},\sigma)|\sigma|^{(n-1)/2}\mathcal{F}_\alpha f(\sigma\mathbf{e}_{\boldsymbol{\theta}})(1+|\sigma|^2)^\gamma\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}})\\ &= \int_{S^{n-1}}\int_0^\infty \mathcal{F}_{\alpha,s}g(\mathbf{e}_{\boldsymbol{\theta}},\sigma)|\sigma|^{(n-1)/2}\mathcal{F}_\alpha f(\sigma\mathbf{e}_{\boldsymbol{\theta}})(1+|\sigma|^2)^\gamma\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}}),\end{aligned}$$

since $\mathcal{F}_{\alpha,s}g$ is even by the evenness of g. Changing the variables $\sigma\mathbf{e}_{\boldsymbol{\theta}} \to \boldsymbol{\xi}$, we have

$$0 = \int_{\mathbb{R}^{n-1}} \mathcal{F}_{\alpha,s}g\left(\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|},|\boldsymbol{\xi}|\right)|\boldsymbol{\xi}|^{-(n-1)/2}\mathcal{F}_\alpha f(\boldsymbol{\xi})(1+|\boldsymbol{\xi}|^2)^\gamma\,\mathrm{d}\boldsymbol{\xi}.$$

Since $f \in \mathcal{S}(\mathbb{R}^n)$ is arbitrary, by Theorem 2.1 $\mathcal{F}_{\alpha,s}g(\boldsymbol{\xi}/|\boldsymbol{\xi}|,|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^{-(n-1)/2}$ is equal to zero almost everywhere and thus $\mathcal{F}_{\alpha,s}g(\boldsymbol{\xi}/|\boldsymbol{\xi}|,|\boldsymbol{\xi}|)$ and g are equal to zero almost everywhere. ■

Corollary 3.7: *For $\gamma \geq 0$, $0 < |\alpha| < \pi$, and $f \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|f\|_{\alpha,\gamma} \leq 2^{-1/2}|C_\alpha|^{n-1}\|R_\alpha f\|_{\alpha,\gamma+(n-1)/2}.$$

This corollary follows from (3.12) and two definitions (3.6) and (3.11) of I^γ_α and $\|\cdot\|_{\alpha,\gamma}$.

Theorem 3.8: *Let $f \in L^2(\mathbb{R}^n)$ have compact support in the unit ball. For $\gamma \geq 0$ and $0 < |\alpha| < \pi$, there is a constant $C_{\alpha,\gamma,n}$ such that $\|R_\alpha f\|_{\alpha,\gamma+(n-1)/2} \leq C_{\alpha,\gamma,n}\|f\|_{\alpha,\gamma}$.*

Proof: Similar to (3.12), we have

$$\begin{aligned}\|R_\alpha f\|^2_{\alpha,\gamma+(n-1)/2} &= 2|C_\alpha|^{2(1-n)} \int_{\mathbb{R}^n} |\mathcal{F}_\alpha f(\boldsymbol{\xi})|^2 (1+|\boldsymbol{\xi}|^2)^{\gamma+(n-1)/2} |\boldsymbol{\xi}|^{1-n}\,\mathrm{d}\boldsymbol{\xi} \\ &= 2|C_\alpha|^{2(1-n)} \int_{|\boldsymbol{\xi}|\geq 1} + \int_{|\boldsymbol{\xi}|\leq 1} |\mathcal{F}_\alpha f(\boldsymbol{\xi})|^2 (1+|\boldsymbol{\xi}|^2)^{\gamma+(n-1)/2} |\boldsymbol{\xi}|^{1-n}\,\mathrm{d}\boldsymbol{\xi}.\end{aligned} \tag{3.13}$$

The first term is bounded by $2^{(n+1)/2}|C_\alpha|^{2(1-n)}\|f\|^2_{\alpha,\gamma}$, since $|\boldsymbol{\xi}|^2 \geq 2^{-1}(1+|\boldsymbol{\xi}|^2)$. Since

$$|\mathcal{F}_\alpha f(\boldsymbol{\xi})| \leq |C_\alpha| \int_{|\mathbf{x}|<1} |f(\mathbf{x})|\mathrm{d}\mathbf{x} \leq |C_\alpha||S^{n-1}|^{1/2}\|f\|_{\alpha,0} \leq |C_\alpha||S^{n-1}|^{1/2}\|f\|_{\alpha,\gamma},$$

where in the second inequality, we used the Hölder inequality and the Parseval identity (2.2), the second term is estimated by

$$\begin{aligned}&\int_{|\boldsymbol{\xi}|\leq 1} |\mathcal{F}_\alpha f(\boldsymbol{\xi})|^2 (1+|\boldsymbol{\xi}|^2)^{\gamma+(n-1)/2} |\boldsymbol{\xi}|^{1-n}\,\mathrm{d}\boldsymbol{\xi} \\ &\quad\leq |C_\alpha|^2|S^{n-1}| \left(\int_{|\boldsymbol{\xi}|\leq 1} (1+|\boldsymbol{\xi}|^2)^{\gamma+(n-1)/2} |\boldsymbol{\xi}|^{1-n}\mathrm{d}\boldsymbol{\xi} \right) \|f\|^2_{\alpha,\gamma}.\end{aligned}$$ ■

For $\gamma \geq 0$ and $T = \{(\mathbf{e}_{\boldsymbol{\theta}}, \mathbf{y}) : \mathbf{e}_{\boldsymbol{\theta}} \in S^{n-1}, \mathbf{y} \in \mathbf{e}_{\boldsymbol{\theta}}^{\perp}\}$, let $H^\gamma_\alpha(T)$ be the collections of L^2-functions bounded by the following norm:

$$\|g\|^2_{\alpha,\gamma} = \int_{S^{n-1}} \int_{\mathbf{e}_{\boldsymbol{\theta}}^{\perp}} \mathcal{F}_{\alpha,\mathbf{y}} g(\mathbf{e}_{\boldsymbol{\theta}}, \boldsymbol{\eta})|^2 (1+|\boldsymbol{\eta}|^2)^\gamma\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}}).$$

Theorem 3.9: *For each $\gamma \geq 0$ and $0 < |\alpha| < \pi$, there exist positive constants $c_{\alpha,\gamma,n}$ and $C_{\alpha,\gamma,n}$ such that for $f \in C^\infty(\mathbb{R}^n)$ with compact support in the unit ball,*

$$c_{\alpha,\gamma,n}\|f\|_{\alpha,\gamma} \leq \|X_\alpha f\|_{\alpha,\gamma+1/2} \leq C_{\alpha,\gamma,n}\|f\|_{\alpha,\gamma}.$$

Proof: We obtain

$$\begin{aligned}\|X_\alpha f\|^2_{\alpha,\gamma} &= \int_{S^{n-1}} \int_{\mathbf{e}_{\boldsymbol{\theta}}^{\perp}} |\mathcal{F}_{\alpha,\mathbf{y}}(X_\alpha f)(\mathbf{e}_{\boldsymbol{\theta}}, \boldsymbol{\eta})|^2 (1+|\boldsymbol{\eta}|^2)^\gamma\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}}) \\ &= (C_\alpha)^{-2} \int_{S^{n-1}} \int_{\mathbf{e}_{\boldsymbol{\theta}}^{\perp}} |\mathcal{F}_\alpha f(\boldsymbol{\eta})|^2 (1+|\boldsymbol{\eta}|^2)^\gamma\,\mathrm{d}\boldsymbol{\eta}\,\mathrm{d}S(\mathbf{e}_{\boldsymbol{\theta}}) \\ &= (C_\alpha)^{-2}|S^{n-2}| \int_{\mathbb{R}^n} |\mathcal{F}_\alpha f(\boldsymbol{\xi})|^2 (1+|\boldsymbol{\xi}|^2)^\gamma |\boldsymbol{\xi}|^{-1}\,\mathrm{d}\mathbf{x} \\ &= (C_\alpha)^{-2}|S^{n-2}|\,\||I^{1/2} f\|^2_{\alpha,\gamma},\end{aligned} \tag{3.14}$$

where in the last line, we used the following equality: (please see [17, Equation (2.8) on the p.190])

$$\int_{S^{n-1}} \int_{\mathbf{e}_{\boldsymbol{\theta}}^{\perp}} f(\mathbf{y})\,\mathrm{d}\mathbf{y}\,\mathrm{d}\mathbf{e}_{\boldsymbol{\theta}} = |S^{n-2}| \int_{\mathbb{R}^n} |\mathbf{x}|^{-1} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

From here on we proceed exactly in the same way in the proof of Theorem 5.1 in [17] . ■

3.3. The partial data problems and range description

Like the regular Radon transform, we have the following support theorem, which follows from Theorem 3.5.

Corollary 3.10: *Let $f \in C^\infty(\mathbb{R}^n)$ have compact support and $0 < |\alpha| < \pi$. If $R_\alpha f(\mathbf{e}_{\boldsymbol\theta}, s) = 0$ for $|s| > M$, then $f(\mathbf{x}) = 0$ for $|\mathbf{x}| > M$.*

Combining (3.2) and Theorem 3.4 in Chapter II in [17], we have the following theorem which similar to Theorem 3.4 in Chapter II in [17].

Proposition 3.11: *Let $0 < |\alpha| < \pi$ and $A \subset S^{n-1}$ be a set of directions such that no non-trivial homogeneous polynomial vanishes on A. If $f \in C^\infty(\mathbb{R}^n)$ with compact support and $R_\alpha f(\mathbf{e}_{\boldsymbol\theta}, s) = 0$ for $\mathbf{e}_{\boldsymbol\theta} \in A$, then $f = 0$.*

Now we study the reconstruction problem for limited data similar to [15]. Let $E \subset S^{n-1}$ be the open set symmetric with respect to the origin. We define the wedges and the projection operator by $w_E := \mathbb{R} \cdot E = \{\sigma \mathbf{e}_{\boldsymbol\theta} : \mathbf{e}_{\boldsymbol\theta} \in E, \sigma \in \mathbb{R}\}$ and $P_{E,\alpha} f = \mathcal{F}_{-\alpha}(\chi_{w_E} \mathcal{F}_\alpha f)$. Here χ_{w_E} is the characteristic function of a set w_E.

Theorem 3.12: *Let $f \in C^\infty(\mathbb{R}^n)$ have compact support and $0 < |\alpha| < \pi$. We have*

$$P_{E,\alpha} f(\mathbf{x}) = 2^{-1}(2\pi)^{1-n} |\csc\alpha|^{n-1} I_\alpha^{-\gamma}[R_\alpha^\# I_\alpha^{\gamma-n+1}(R_{\alpha,E} f)](\mathbf{x}),$$

where $R_{\alpha,E} f(\mathbf{e}_{\boldsymbol\theta}, s) = \chi_E(\mathbf{e}_{\boldsymbol\theta}) R_\alpha f(\mathbf{e}_{\boldsymbol\theta}, s)$.

Proof: By Theorem 3.1, we have

$$\begin{aligned} \mathcal{F}_\alpha(I_\alpha^\gamma P_{E,\alpha} f)(\boldsymbol\xi) &= |\boldsymbol\xi|^{-\gamma} \chi_{w_E}(\boldsymbol\xi) \mathcal{F}_\alpha f(\boldsymbol\xi) \\ &= (C_\alpha)^{n-1} |\boldsymbol\xi|^{-\gamma} \chi_E\left(\frac{\boldsymbol\xi}{|\boldsymbol\xi|}\right) \mathcal{F}_\alpha(R_\alpha f)\left(\frac{\boldsymbol\xi}{|\boldsymbol\xi|}, |\boldsymbol\xi|\right) \\ &= (C_\alpha)^{n-1} |\boldsymbol\xi|^{-\gamma} \mathcal{F}_\alpha(R_{\alpha,E} f)\left(\frac{\boldsymbol\xi}{|\boldsymbol\xi|}, |\boldsymbol\xi|\right). \end{aligned} \tag{3.15}$$

Together with the inverse fractional Fourier transform, we have

$$\begin{aligned} I_\alpha^\gamma P_{E,\alpha} f(\mathbf{x}) &= (C_{-\alpha})^n \int_{\mathbb{R}^n} \mathcal{F}_\alpha(P_{E,\alpha} I_\alpha^\gamma f)(\boldsymbol\xi)\, \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2 + |\boldsymbol\xi|^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\boldsymbol\xi \csc\alpha}\, \mathrm{d}\boldsymbol\xi \\ &= (C_{-\alpha})^n \int_{S^{n-1}} \int_0^\infty \mathcal{F}_\alpha(P_{E,\alpha} I_\alpha^\gamma f)(\sigma \mathbf{e}_{\boldsymbol\theta})\, \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2 + \sigma^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\sigma \mathbf{e}_{\boldsymbol\theta} \csc\alpha} |\sigma|^{n-1}\, \mathrm{d}\sigma\, \mathrm{d}S(\mathbf{e}_{\boldsymbol\theta}), \end{aligned}$$

where in the last line, we changed the variables $\boldsymbol\xi \to \sigma \mathbf{e}_{\boldsymbol\theta}$. Now the integrated function is even with respect to $(\sigma, \mathbf{e}_{\boldsymbol\theta})$, that is, $H(\sigma, \mathbf{e}_{\boldsymbol\theta}) = H(-\sigma, -\mathbf{e}_{\boldsymbol\theta})$ when H is the integrated function. Since for an even function H,

$$\int_{S^{n-1}}\int_0^\infty H(\sigma,\mathbf{e}_\theta)\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_\theta)=\int_{S^{n-1}}\int_{-\infty}^0 H(-\sigma,-\mathbf{e}_\theta)\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_\theta),$$

we have

$$\begin{aligned}
&I_\alpha^\gamma P_{E,\alpha}f(\mathbf{x})\\
&=2^{-1}(C_{-\alpha})^n\int_{S^{n-1}}\int_{\mathbb{R}}\mathcal{F}_\alpha(P_{E,\alpha}I_\alpha^\gamma f)(\sigma\mathbf{e}_\theta)\,\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha+\mathrm{i}\mathbf{x}\cdot\sigma\mathbf{e}_\theta\csc\alpha}|\sigma|^{n-1}\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_\theta)\\
&=2^{-1}(C_{-\alpha})^n(C_\alpha)^{n-1}\int_{S^{n-1}}\int_{\mathbb{R}}|\sigma|^{-\gamma+n-1}\mathcal{F}_\alpha(R_{\alpha,E}f)(\mathbf{e}_\theta,\sigma)\\
&\quad\times\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha+\mathrm{i}\mathbf{x}\cdot\sigma\mathbf{e}_\theta\csc\alpha}\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_\theta)\\
&=2^{-1}(2\pi)^{1-n}|\csc\alpha|^{n-1}\int_{S^{n-1}}I_\alpha^{\gamma-n+1}(R_{\alpha,E}f)(\mathbf{e}_\theta,\mathbf{x}\cdot\mathbf{e}_\theta)\,\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2-(\mathbf{x}\cdot\mathbf{e}_\theta)^2)\cot\alpha}\,\mathrm{d}S(\mathbf{e}_\theta),
\end{aligned}$$

where in the third line, we used (3.15). ■

Now we describe the range of the fractional Radon transform using the range description of the regular Radon transform.

Theorem 3.13: *Let $f\in\mathcal{S}(\mathbb{R}^n)$ and $0<|\alpha|<\pi$. If $g\in\mathcal{S}(S^{n-1}\times\mathbb{R})$ be even (i.e. $g(\mathbf{e}_\theta,s)=g(-\mathbf{e}_\theta,-s)$), then for $m=0,1,2,\ldots$,*

$$\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha}R_\alpha f(\mathbf{e}_\theta,s)s^m\,\mathrm{d}s$$

is a homogeneous polynomials of degree m. Also, if for each $m=0,1,2,\ldots$,

$$\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha}g(\mathbf{e}_\theta,s)s^m\,\mathrm{d}s$$

is a homogeneous polynomials of degree m, then there is $f\in\mathcal{S}(\mathbb{R}^n)$ such that $g=R_\alpha f$.

Proof: We compute

$$\begin{aligned}
\int_{\mathbb{R}}\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha}R_\alpha f(\mathbf{e}_\theta,s)s^m\,\mathrm{d}s&=\int_{\mathbb{R}}\int_{\mathbf{e}_\theta^\perp}f(s\mathbf{e}_\theta+\boldsymbol{\tau})\,\mathrm{e}^{\mathrm{i}2^{-1}(|\boldsymbol{\tau}|^2+s^2)\cot\alpha}s^m\,\mathrm{d}\boldsymbol{\tau}\,\mathrm{d}s\\
&=\int_{\mathbb{R}^n}f(\mathbf{x})\,\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha}(\mathbf{x}\cdot\mathbf{e}_\theta)^m\,\mathrm{d}\mathbf{x},
\end{aligned}$$

where we put $\mathbf{x}=s\mathbf{e}_\theta+\boldsymbol{\tau}$.

On the other hand, by the range description of the regular Radon transform R derived in [17, Theorem 4.2 in Chapter II], there is $F\in\mathcal{S}(\mathbb{R}^n)$ such that $\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha}g=RF$. Setting $f(\mathbf{x})=\mathrm{e}^{-\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha}F(\mathbf{x})$ gives $f\in\mathcal{S}(\mathbb{R}^n)$ and $\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha}g(\mathbf{e}_\theta,s)=R[\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha}f(\mathbf{x})](\mathbf{e}_\theta,s)$ which is equivalent to $g=R_\alpha f$ by (3.2). ■

4. The fractional exponential Radon transform

For $f \in \mathcal{S}(\mathbb{R}^2)$ with compact support, the exponential Radon transform is defined by

$$T_\mu f(\mathbf{e}_\theta, s) = \int_{\mathbb{R}^2} \mathrm{e}^{\mu \mathbf{x}\cdot\mathbf{e}_\theta^\perp} f(\mathbf{x})\delta(\mathbf{x}\cdot\mathbf{e}_\theta - s)\,\mathrm{d}\mathbf{x} = \int_{\mathbb{R}} \mathrm{e}^{\mu\tau} f(s\mathbf{e}_\theta + \tau\mathbf{e}_\theta^\perp)\,\mathrm{d}\tau,$$

where θ is the polar angle of the unit vector $\mathbf{e}_\theta \in S^1$, that is, $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$, and $\mathbf{e}_\theta^\perp = (-\sin\theta, \cos\theta)$ and μ is a constant. Similar to definition (3.1) of the fractional Radon transform, we can define the fractional exponential Radon transform $T_{\mu,\alpha} f$, $0 < |\alpha| < \pi$ by

$$\begin{aligned} T_{\mu,\alpha} f(\mathbf{e}_\theta, s) &= \int_{\mathbf{e}_\theta^\perp} f(s\mathbf{e}_\theta + \tau\mathbf{e}_\theta^\perp)\,\mathrm{e}^{\mathrm{i}2^{-1}(|\tau|^2-\mu^2)\cot\alpha}\,\mathrm{e}^{\mu\tau\csc\alpha}\,\mathrm{d}\tau \\ &= \int_{\mathbb{R}^2} f(\mathbf{x})\,\mathrm{e}^{\mathrm{i}2^{-1}(|\mathbf{x}|^2-s^2-\mu^2)\cot\alpha}\,\delta(\mathbf{e}_\theta\cdot\mathbf{x}-s)\,\mathrm{e}^{\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha}\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Like $T_\mu f$, $T_{\mu,\alpha} f$ is not even. Also, we notice that $T_\mu f = T_{\mu,\pi/2} f$ and

$$T_{\mu,\alpha} f(\mathbf{e}_\theta, s) = \mathrm{e}^{-\mathrm{i}2^{-1}(s^2+\mu^2)\cot\alpha}\,T_{\mu\csc\alpha}(\mathrm{e}^{\mathrm{i}2^{-1}|\mathbf{x}|^2\cot\alpha} f(\mathbf{x}))(\mathbf{e}_\theta, s). \tag{4.1}$$

Now we have the analogue of the Fourier slice theorem:

Theorem 4.1: *For $f \in \mathcal{S}(\mathbb{R}^2)$ with compact support and $0 < |\alpha| < \pi$, we have*

$$\mathcal{F}_{\alpha,s}(T_{\mu,\alpha} f)(\mathbf{e}_\theta, \sigma) = (C_\alpha)^{-1}\mathcal{F}_\alpha f(\sigma\mathbf{e}_\theta + \mathrm{i}\mu\mathbf{e}_\theta^\perp).$$

Proof: Taking the fractional Fourier transform of $T_{\mu,\alpha} f$ with respect to s, we have

$$\begin{aligned} \mathcal{F}_{\alpha,s}(T_{\mu,\alpha} f)(\mathbf{e}_\theta, \sigma) &= C_\alpha \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2^{-1}(\sigma^2+s^2)\cot\alpha - \mathrm{i}s\sigma\csc\alpha} \int_{\mathbb{R}^2} f(\mathbf{x})\,\mathrm{e}^{\mathrm{i}2^{-1}(|\mathbf{x}|^2-s^2-\mu^2)\cot\alpha} \\ &\quad \times \mathrm{e}^{\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha}\,\delta(\mathbf{e}_\theta\cdot\mathbf{x}-s)\,\mathrm{d}\mathbf{x}\,\mathrm{d}s \\ &= C_\alpha \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{i}2^{-1}(\sigma^2-\mu^2+|\mathbf{x}|^2)\cot\alpha - \mathrm{i}\mathbf{x}\cdot\mathbf{e}_\theta\sigma\csc\alpha}\,\mathrm{e}^{\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha} f(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= (C_\alpha)^{-1}\mathcal{F}_\alpha f(\sigma\mathbf{e}_\theta + \mathrm{i}\mu\mathbf{e}_\theta^\perp). \qquad \blacksquare \end{aligned}$$

In fact, the fractional exponential Radon transform is defined based on Theorem 4.1 as the fractional Radon transform is defined using Theorem 3.1.

4.1. Inversion formula

To obtain the inversion formula for the fractional exponential Radon transform, we introduce the dual operator $T^{\#}_{\mu,\alpha}$, defined by

$$T^{\#}_{\mu,\alpha} g(\mathbf{x}) = \int_{S^1} \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2-(\mathbf{x}\cdot\mathbf{e}_\theta)^2-\mu^2)\cot\alpha}\,\mathrm{e}^{\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha} g(\mathbf{e}_\theta, \mathbf{x}\cdot\mathbf{e}_\theta)\,\mathrm{d}S(\mathbf{e}_\theta).$$

Then by simple computation, we can easily show

$$\int_{\mathbb{R}}\int_{S^1} T_{\mu,\alpha}f(\mathbf{e}_\theta,s)[g(\mathbf{e}_\theta,s)]^- \mathrm{d}S(\mathbf{e}_\theta)\,\mathrm{d}s = \int_{\mathbb{R}^2} f(\mathbf{x})[T^\#_{\mu,\alpha}g(\mathbf{x})]^- \mathrm{d}\mathbf{x}.$$

Also, like (3.4), we have

$$T^\#_{\mu,\alpha}g(\mathbf{x}) = \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|-\mu^2)\cot\alpha} T^\#_{\mu\csc\alpha}(\mathrm{e}^{\mathrm{i}2^{-1}s^2\cot\alpha}g)(\mathbf{x}),$$

where $T^\#_\mu = T^\#_{\mu,\pi/2}$ is the dual operator to $T_{\mu,\pi/2} = T_\mu$.

Theorem 4.2: *Let $f \in \mathcal{S}(\mathbb{R}^2)$ with compact support and $0 < |\alpha| < \pi$. For $\gamma < 2$, we have*

$$f(\mathbf{x}) = (4\pi)^{-1}|\csc\alpha| I_\alpha^{-\gamma} T^\#_{-\mu,\alpha}[I^{\gamma-1}_{\alpha,\mu} T_{\mu,\alpha}f(\mathbf{e}_\theta,s)](\mathbf{x}), \tag{4.2}$$

where $I^{-\gamma}_{\alpha,\mu}$ is the generalized Riesz potential

$$\mathcal{F}_\alpha(I^{-\gamma}_{\alpha,\mu}h)(\sigma) = \begin{cases} |\sigma|(\sigma^2-\mu^2)^{(\gamma-1)/2}\mathcal{F}_\alpha h(\sigma), & |\sigma| > |\mu|, \\ 0, & \textit{otherwise}. \end{cases}$$

When $\alpha = \pi/2$ and $\gamma = 0$, (4.2) becomes the inversion formula for the exponential Radon transform which is the same as the formula derived in [17, Theorem 6.1 in Chapter II]. To prove this theorem, we need the following lemma:

Lemma 4.3: *Let $h : \mathbb{C} \to \mathbb{C}$ be an analytic function with $h(z) = h(z + 2\pi)$. Then we have for a constant $b \in \mathbf{R}$*

$$\int_0^{2\pi} h(\theta)\,\mathrm{d}\theta = \int_0^{2\pi} h(\theta + \mathrm{i}b)\,\mathrm{d}\theta.$$

Proof: By Cauchy's integral theorem we have

$$\int_C h(z)\,\mathrm{d}z = 0, \tag{4.3}$$

where C is a simple closed curve in $\mathbb{C}$. Let us define $C : [0,4] \to \mathbb{C}$ by

$$C(\lambda) = \begin{cases} 2\pi\lambda & \text{if } 0 < \lambda < 1, \\ 2\pi + b(\lambda-1)\mathrm{i} & \text{if } 1 < \lambda < 2, \\ 2\pi(3-\lambda) + b\mathrm{i} & \text{if } 2 < \lambda < 3, \\ b(4-\lambda)\mathrm{i} & \text{if } 3 < \lambda < 4. \end{cases}$$

By 2π-periodicity of h, we have

$$\begin{aligned} \int_{C[1,2]} h(z)\,\mathrm{d}z &= b\mathrm{i}\int_1^2 h(2\pi + b(\lambda-1)\mathrm{i})\,\mathrm{d}\lambda = b\mathrm{i}\int_3^4 h(b(4-\lambda)\mathrm{i})\,\mathrm{d}\lambda \\ &= -\int_{C[3,4]} h(z)\,\mathrm{d}z. \end{aligned}$$

Together with (4.3), we have

$$\int_0^1 h(2\pi\lambda)\,\mathrm{d}\lambda = (2\pi)^{-1}\int_{C[0,1]} h(z)\,\mathrm{d}z = -(2\pi)^{-1},$$
$$\int_{C[2,3]} h(z)\,\mathrm{d}z = \int_2^3 h(2\pi(3-\lambda)+b\mathrm{i})\,\mathrm{d}\lambda,$$

which is equivalent to our assertion. ■

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2: Let us start with the inverse fractional Fourier transform:

$$\begin{aligned} I_\alpha^\gamma f(\mathbf{x}) &= (C_{-\alpha})^2\int_{\mathbb{R}^2}|\boldsymbol{\xi}|^{-\gamma}\mathcal{F}_\alpha f(\boldsymbol{\xi})\,\mathrm{e}^{\mathrm{i}\boldsymbol{\xi}\cdot\mathbf{x}\csc\alpha}\,\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+|\boldsymbol{\xi}|^2)\cot\alpha}\,\mathrm{d}\boldsymbol{\xi}\\ &= (C_{-\alpha})^2\int_0^\infty\int_{S^1}\mathcal{F}_\alpha f(\sigma\mathbf{e}_\theta)\,\mathrm{e}^{\mathrm{i}\sigma\mathbf{e}_\theta\cdot\mathbf{x}\csc\alpha}\,\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha}\sigma^{1-\gamma}\,\mathrm{d}S(\mathbf{e}_\theta)\,\mathrm{d}\sigma, \end{aligned}$$

where in the last line, we changed the variables $\boldsymbol{\xi}\to\sigma\mathbf{e}_\theta$. As in the proof of Theorem 3.12, an integrand function is even with respect to $(\sigma,\mathbf{e}_\theta)$ and thus we have

$$\begin{aligned} I_\alpha^\gamma f(\mathbf{x}) &= 2^{-1}(C_{-\alpha})^2\int_{\mathbb{R}}\int_{S^1}\mathcal{F}_\alpha f(\sigma\mathbf{e}_\theta)\,\mathrm{e}^{\mathrm{i}\sigma\mathbf{e}_\theta\cdot\mathbf{x}\csc\alpha}\,\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha}|\sigma|^{1-\gamma}\,\mathrm{d}S(\mathbf{e}_\theta)\,\mathrm{d}\sigma\\ &= 2^{-1}(C_{-\alpha})^2\int_{|\sigma|>|\mu|}\int_{S^1}\mathcal{F}_\alpha f(\sqrt{\sigma^2-\mu^2}\mathbf{e}_\theta)\,\mathrm{e}^{\mathrm{i}\sqrt{\sigma^2-\mu^2}\mathbf{e}_\theta\cdot\mathbf{x}\csc\alpha}\\ &\quad\times\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2-\mu^2)\cot\alpha}|\sigma|(\sigma^2-\mu^2)^{-\gamma/2}\,\mathrm{d}S(\mathbf{e}_\theta)\,\mathrm{d}\sigma, \end{aligned}\tag{4.4}$$

where in the last line, we changed the variables $\sigma\to\sqrt{\sigma^2-\mu^2}$. Let $\phi_{\sigma,\mu}=(\mathrm{i}/2)\ln[(\sigma+\mu)/(\sigma-\mu)]$. Applying Lemma 4.3 to (4.4) and Theorem 4.1, $I_\alpha^\gamma f(\mathbf{x})$ becomes to be equal to

$$\begin{aligned} &2^{-1}(C_{-\alpha})^2\int_{|\sigma|>|\mu|}\int_{S^1}\mathcal{F}_\alpha f((\sqrt{\sigma^2-\mu^2}\mathbf{e}_{\theta+\phi_{\sigma,\mu}})\,\mathrm{e}^{\mathrm{i}\sqrt{\sigma^2-\mu^2}\mathbf{e}_{\theta+\phi_{\sigma,\mu}}\cdot\mathbf{x}\csc\alpha}\\ &\qquad\times\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2-\mu^2)\cot\alpha}|\sigma|(\sigma^2-\mu^2)^{-\gamma/2}\,\mathrm{d}S(\mathbf{e}_\theta)\,\mathrm{d}\sigma\\ &\quad=\frac{C_{-\alpha}}{4\pi|\sin\alpha|}\int_{S^1}\int_{|\sigma|>|\mu|}\mathcal{F}_{\alpha,s}(T_{\mu,\alpha}f)(\mathbf{e}_\theta,\sigma)\,\mathrm{e}^{\mathrm{i}\sigma\mathbf{e}_\theta\cdot\mathbf{x}\csc\alpha-\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha}\\ &\qquad\times\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2-\mu^2)\cot\alpha}|\sigma|(\sigma^2-\mu^2)^{-\gamma/2}\,\mathrm{d}\sigma\,\mathrm{d}S(\mathbf{e}_\theta), \end{aligned}\tag{4.5}$$

since $\sqrt{\sigma^2-\mu^2}(\cos(\theta+\phi_{\sigma,\mu}),\sin(\theta+\phi_{\sigma,\mu}))=\sigma\mathbf{e}_\theta+\mathrm{i}\mu\mathbf{e}_\theta^\perp$. By the definition of $I_{\alpha,\mu}^{-\gamma}$, (4.5) becomes

$$\begin{aligned} I_\alpha^\gamma f(\mathbf{x}) &= (4\pi)^{-1}|\csc\alpha|\int_{S^1}I_{\alpha,\mu}^{\gamma-1}T_{\mu,\alpha}f(\mathbf{e}_\theta,\mathbf{e}_\theta\cdot\mathbf{x})\\ &\qquad\mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2-(\mathbf{e}_\theta\cdot\mathbf{x})^2-\mu^2)\cot\alpha}\,\mathrm{e}^{-\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha}\,\mathrm{d}S(\mathbf{e}_\theta). \end{aligned}$$ ■

4.2. Stability estimates

In this subsection, we show that the problem of reconstructing from the fractional exponential Radon transform is well-posed in the following sense: if f satisfying $g = T_{\mu,\alpha}f$ is uniquely determined for any g belonging to a certain space, the function f depends continuously on g.

Theorem 4.4: *For $\gamma \geq 0$, $0 < |\alpha| < \pi$, and $f \in \mathcal{S}(\mathbb{R}^2)$, we have*

$$\|f\|_{\alpha,\gamma} \leq |C_\alpha| \, \|T_{\mu,\alpha}f\|_{\alpha,\gamma+1/2}.$$

When $\alpha = \pi/2$, we get the stability estimate of the regular exponential Radon transform.

Proof: Notice that from Theorem 4.1, we have

$$\begin{aligned}\mathcal{F}_{\alpha,s}(T_{\mu,\alpha}f)(\mathbf{e}_\theta, \sqrt{\sigma^2+\mu^2}) &= (C_\alpha)^{-1}\mathcal{F}_\alpha f(\sqrt{\sigma^2+\mu^2}\mathbf{e}_\theta + \mathrm{i}\mu\mathbf{e}_\theta^\perp) \\ &= (C_\alpha)^{-1}\mathcal{F}_\alpha f(\sigma\mathbf{e}_{\theta+\phi_{\sqrt{\sigma^2+\mu^2},\mu}}),\end{aligned} \tag{4.6}$$

where $\phi_{\sqrt{\sigma^2+\mu^2},\mu} = (\mathrm{i}/2)\ln[(\sqrt{\sigma^2+\mu^2}+\mu)/(\sqrt{\sigma}^2+\mu^2-\mu)] = (\mathrm{i}/2)\ln[\sigma/(\sigma^2 + 2\mu^2 - 2\mu\sqrt{\sigma^2+\mu^2})]$.

Similar to (3.12), let us consider $\|f\|^2_{\alpha,\gamma}$:

$$\begin{aligned}\|f\|^2_{\alpha,\gamma} &= \int_0^{2\pi}\int_0^\infty |\mathcal{F}_\alpha f(\sigma\mathbf{e}_{\theta+\phi_{\sqrt{\sigma^2+\mu^2},\mu}})|^2(1+|\sigma|^2)^\gamma|\sigma|\,\mathrm{d}\sigma\,\mathrm{d}\theta \\ &= |C_\alpha|^2\int_0^{2\pi}\int_0^\infty |\mathcal{F}_{\alpha,s}(T_{\mu,\alpha}f)(\mathbf{e}_\theta,\sqrt{\sigma^2+\mu^2})|^2(1+|\sigma|^2)^\gamma|\sigma|\,\mathrm{d}\sigma\,\mathrm{d}\theta \\ &= |C_\alpha|^2\int_0^{2\pi}\int_{|\mu|}^\infty |\mathcal{F}_{\alpha,s}(T_{\mu,\alpha}f)(\mathbf{e}_\theta,\sigma)|^2(1+|\sqrt{\sigma^2-\mu^2}|^2)^\gamma|\sigma|\,\mathrm{d}\sigma\,\mathrm{d}\theta \\ &\leq |C_\alpha|^2\int_0^{2\pi}\int_{\mathbb{R}} |\mathcal{F}_{\alpha,s}(T_{\mu,\alpha}f)(\mathbf{e}_\theta,\sigma)|^2(1+|\sigma|^2)^{\gamma+1/2}\,\mathrm{d}\sigma\,\mathrm{d}\theta \\ &= |C_\alpha|^2\|T_{\mu,\alpha}f\|^2_{\alpha,\gamma+1/2},\end{aligned}$$

where in the first and second lines, we used Lemma 4.3 and (4.6), respectively, and in the third line, we changed the variables $\sqrt{\sigma^2+\mu^2} \to \sigma$. ■

4.3. The partial data problem

In this subsection, we study the fractional exponential Radon transform version of Theorem 3.12. Let $E \subset S^1$ be a set. As in Section 3.3, we define the projection operator by

$$\mathcal{L}_{E,\alpha}f = \mathcal{F}_{-\alpha}(\chi_E(\mathbf{e}_{\theta-\phi_{\sqrt{\sigma^2+\mu^2},\mu}})\mathcal{F}_\alpha f(\sigma\mathbf{e}_\theta)).$$

Here χ_E is the characteristic function of a set E, again.

Theorem 4.5: *Let $f \in C^\infty(\mathbb{R}^2)$ have compact support and $0 < |\alpha| < \pi$. We have*

$$\mathcal{L}_{E,\alpha} f(\mathbf{x}) = (4\pi)^{-1} |\csc\alpha| I_\alpha^{-\gamma} [T^{\#}_{-\mu,\alpha} I_\alpha^{\gamma-1} (T_{\mu,\alpha,E} f)](\mathbf{x}),$$

where $T_{\mu,\alpha,E} f(\mathbf{e}_\theta, s) = \chi_E(\mathbf{e}_\theta) T_{\mu,\alpha} f(\mathbf{e}_\theta, s)$.

Proof: By (4.6), we have

$$\begin{aligned}\chi_E(\mathbf{e}_\theta)\mathcal{F}_\alpha f(\sigma \mathbf{e}_{\theta+\phi_{\sqrt{\sigma^2+\mu^2},\mu}}) &= C_\alpha \chi_E(\mathbf{e}_\theta) \mathcal{F}_\alpha (T_{\mu,\alpha} f)(\mathbf{e}_\theta, \sqrt{\sigma^2+\mu^2}) \\ &= C_\alpha \mathcal{F}_\alpha (T_{\mu,\alpha,E} f)(\mathbf{e}_\theta, \sqrt{\sigma^2+\mu^2}). \end{aligned} \tag{4.7}$$

Together with the inverse fractional Fourier transform, we have to

$$\begin{aligned} I_\alpha^\gamma \mathcal{L}_{E,\alpha} f(\mathbf{x}) &= (C_{-\alpha})^2 \int_{\mathbb{R}^2} \mathcal{F}_\alpha (I_\alpha^\gamma \mathcal{L}_{E,\alpha} f)(\boldsymbol{\xi})\, \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+|\boldsymbol{\xi}|^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}\csc\alpha}\, \mathrm{d}\boldsymbol{\xi} \\ &= (C_{-\alpha})^2 \int_{S^1}\int_0^\infty \mathcal{F}_\alpha (I_\alpha^\gamma \mathcal{L}_{E,\alpha} f)(\sigma \mathbf{e}_\theta) \\ &\quad \times \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\sigma\mathbf{e}_\theta\csc\alpha} |\sigma|\, \mathrm{d}\sigma\, \mathrm{d}S(\mathbf{e}_\theta), \end{aligned}$$

where in the second equality, we changed the variables $\boldsymbol{\xi} \to \sigma \mathbf{e}_\theta$. As in the proof of Theorem 3.12, an integrand function is even with respect to $(\sigma, \mathbf{e}_\theta)$ and thus we have

$$\begin{aligned} I_\alpha^\gamma \mathcal{L}_{E,\alpha} f(\mathbf{x}) &= 2^{-1}(C_{-\alpha})^2 \int_{S^1}\int_{\mathbb{R}} \mathcal{F}_\alpha f(\sigma\mathbf{e}_\theta) \chi_E(\mathbf{e}_{\theta-\phi_{\sqrt{\sigma^2+\mu^2},\mu}}) \\ &\quad \times \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\sigma\mathbf{e}_\theta\csc\alpha} |\sigma|^{1-\gamma}\, \mathrm{d}\sigma\, \mathrm{d}S(\mathbf{e}_\theta) \\ &= 2^{-1}(C_{-\alpha})^2 \int_{S^1}\int_{\mathbb{R}} \mathcal{F}_\alpha f(\sigma \mathbf{e}_{\theta+\phi_{\sqrt{\sigma^2+\mu^2},\mu}}) \chi_E(\mathbf{e}_\theta) \\ &\quad \times \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\sigma\mathbf{e}_{\theta+\phi_{\sqrt{\sigma^2+\mu^2},\mu}}\csc\alpha} |\sigma|^{1-\gamma}\, \mathrm{d}\sigma\, \mathrm{d}S(\mathbf{e}_\theta), \end{aligned}$$

where in the last equality, we used Lemma 4.3. Together with (4.7), we have

$$\begin{aligned} I_\alpha^\gamma \mathcal{L}_{E,\alpha} f(\mathbf{x}) &= 2^{-1}(C_{-\alpha})^2 C_\alpha \int_{S^1}\int_{\mathbb{R}} |\sigma|^{1-\gamma} \mathcal{F}_\alpha (T_{\mu,\alpha,E} f)(\mathbf{e}_\theta, \sqrt{\sigma^2+\mu^2}) \\ &\quad \times \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\sigma\mathbf{e}_{\theta+\phi_{\sqrt{\sigma^2+\mu^2},\mu}}\csc\alpha}\, \mathrm{d}\sigma\, \mathrm{d}S(\mathbf{e}_\theta) \\ &= 2^{-1}(C_{-\alpha})^2 C_\alpha \int_{S^1}\int_{|\sigma|>|\mu|} |\sigma|(\sigma^2-\mu^2)^{-\gamma/2} \mathcal{F}_\alpha (T_{\mu,\alpha,E} f)(\mathbf{e}_\theta, \sigma) \\ &\quad \times \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2+\sigma^2-\mu^2)\cot\alpha + \mathrm{i}\mathbf{x}\cdot\sqrt{\sigma^2-\mu^2}\mathbf{e}_{\theta+\phi_{\sigma,\mu}}\csc\alpha}\, \mathrm{d}\sigma\, \mathrm{d}S(\mathbf{e}_\theta) \\ &= (4\pi)^{-1}|\csc\alpha| \int_{S^1} I_{\mu,\alpha}^{\gamma-1}(T_{\mu,\alpha,E} f)(\mathbf{e}_\theta, \mathbf{x}\cdot\mathbf{e}_\theta)\, \mathrm{e}^{-\mathrm{i}2^{-1}(|\mathbf{x}|^2-(\mathbf{x}\cdot\mathbf{e}_\theta)^2-\mu^2)\cot\alpha} \\ &\quad \times \mathrm{e}^{-\mu\mathbf{x}\cdot\mathbf{e}_\theta^\perp\csc\alpha}\, \mathrm{d}S(\mathbf{e}_\theta), \end{aligned}$$

where in third line, we used the change of variables $\sqrt{\sigma^2+\mu^2} \to \sigma$ and in the last line, we used the identity $\sqrt{\sigma^2-\mu^2}\mathbf{e}_{\theta+\phi_{\sigma,\mu}} = \sigma\mathbf{e}_\theta + \mathrm{i}\mu\mathbf{e}_\theta^\perp$. ■

Acknowledgments

The author would like to thank the referees for the helpful suggestions.

Disclosure statement

No potential conflict of interest was reported by the author.

Funding

This work was supported by the National Research Foundation of Korea grant funded by the Korea government (MSIP) (2015R1C1A1A01051674) and the TJ Park Science Fellowship of POSCO TJ Park Foundation.

References

[1] McBride AC, Kerr FH. On Namias's fractional Fourier transforms. IMA J Appl Math. 1987;39(2):159–175.
[2] Namias V. The fractional order Fourier transform and its application to quantum mechanics. IMA J Appl Math. 1980;25(3):241–265.
[3] Mendlovic D, Ozaktas HM. Fractional Fourier transforms and their optical implementation: I. J Opt Soc Am A. 1993;10(9):1875–1881.
[4] Kerr FH. Namias' fractional Fourier transforms on L^2 and applications to differential equations. J Math Anal Appl. 1988;136(2):404–418.
[5] Lohmann AW. Image rotation, Wigner rotation, and the fractional Fourier transform. J Opt Soc Am A. 1993;10(10):2181–2186.
[6] Mendlovic D, Ozaktas HM, Lohmann AW. Graded-index fibers, Wigner-distribution functions, and the fractional Fourier transform. Appl Opt. 1994;33(26):6188–6193.
[7] Ozaktas HM, Kutay MA. The fractional Fourier transform. European Control Conference (ECC); 2001 Sep 4–7; Porto, Portugal.
[8] Ozaktas HM, Mendlovic D. Fractional Fourier transforms and their optical implementation. II. J Opt Soc Am A. 1993;10(12):2522–2531.
[9] Ozaktas HM, Kutay MA, Zalevsky Z. The fractional Fourier transform: with applications in optics and signal processing. Chichester: Wiley; 2001. (Wiley Series in Pure and Applied Optics).
[10] Saavedra G, Furlan WD. The Radon–Wigner transform and its application to first-order optical systems. Frontiers in Optics (Optical Society of America); 2009 Oct 11–15; San Jose, CA; 2009 page FWW2.
[11] Parot V, Sing-Long C, Lizama C, et al. Application of the fractional Fourier transform to image reconstruction in MRI. Magn Reson Med. 2012;68(1):17–29.
[12] Zalevsky Z, Mendlovic D. Fractional Radon transform: definition. Appl Opt. 1996;35(23): 4628–4631.
[13] Zalevsky Z, Mendlovic D. Fractional wiener filter. Appl Opt. 1996;35(20):3930–3936.
[14] Fan H-Y, Chen J-H. Fractional Radon transform and transform of Wigner operator. Commun Theor Phys. 2003;39(2):147–150.
[15] Frikel J, Quinto ET. Characterization and reduction of artifacts in limited angle tomography. Inverse Probl. 2013;29(12):125007.
[16] Helgason S. The Radon transform. Boston (MA): Birkhäuser; 1999.
[17] Natterer F. The mathematics of computerized tomography. Philadelphia (PA): Society for Industrial and Applied Mathematics; 2001.